1. Generating random variables

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1. of interest

out of c components, at least t of them will last h hours.

step 1: sample $X_1, X_2 \dots X_c \overset{iid}{\sim} \exp(\lambda)$

step 2: $Y = \begin{cases} 1 & \text{if at least } t \text{ of them} > h \\ 0 & o/w. \end{cases}$ → of interest p.

$\therefore p = E(Y)$

∴ 重复 step 1 and 2. 得到 n 个 Y.

step 3: 根据 WLLN $\frac{1}{n}\sum Y_i \to E(Y)$

∴ 求得 E(Y), 从而得到 p.

∴ step 1 是要 generate random variables.

那么, 有哪些方法可以用呢.

① probability integral transformation.

X continuous. w/ CDF $F_X(x)$

def $Y = F_X(X)$, then

$Y \sim unif(0,1) \Rightarrow P(Y \le y) = y$.

$$\begin{aligned} P(Y \le y) &= P(F_X(X) \le y) \\ &= P(F_X^{-1}(F_X(X)) \le F_X^{-1}(y)) \\ &= P(X \le F_X^{-1}(y)) \\ &= F_X F_X^{-1}(y) \\ &= y. \end{aligned}$$

$\Rightarrow Y \sim unif(0,1)$

2024-07-09 update

$U = F_Y(Y) \sim Unif(0,1)$

$Y \sim exp(\lambda)$

$F_Y(y) = 1 - e^{-\lambda y}$

$\therefore U = 1 - e^{-\lambda Y} = F_Y(Y)$

$\therefore e^{-\lambda Y} = 1 - U$

$Y = -\frac{1}{\lambda}\log(1-U) = F_Y^{-1}(U)$

所以, Y 有 CDF $F_Y(y)$, $Y \sim \exp(\lambda)$

$F_Y(Y) = U \Rightarrow \therefore U \sim unif(0,1)$

$$\begin{aligned} \therefore P(U \le u) &= P(F_Y(Y) \le u) \\ &= P(Y \le F_Y^{-1}(u)) \\ &= u. \end{aligned}$$

$\therefore Y = F_Y^{-1}(U) \sim F_Y$

$$\begin{aligned} P(Y \le y) &= P(F_Y^{-1}(U) \le y) \\ &= P(U \le F_Y(y)) \\ &= F_Y(y) \end{aligned}$$

$\Rightarrow \boxed{\begin{aligned} F_Y(Y) &= U \\ Y &= F_Y^{-1}(U) \end{aligned}}$

$f_Y(y) = \frac{1}{\lambda} e^{-\frac{1}{\lambda}t}$

$F_Y(y) = \int_0^t \frac{1}{\lambda} e^{-\frac{1}{\lambda}t} dt$

$-e^{-\frac{1}{\lambda}t}\Big|_0^t = 1 - e^{-\frac{1}{\lambda}t}$

parametrization 方法不一样, 结果会不一样

∴ 求 F_Y^{-1}

$y = 1 - e^{-\frac{1}{\lambda}t}$

$1 - y = e^{-\frac{1}{\lambda}t}$

$\log(1-y) = -\frac{1}{\lambda}t$

$-\lambda\log(1-y) = t(y)$

$Y = -\lambda\log(1-u) \sim \exp(\lambda)$

∴ 这样, step 1 就 decompose 为先产生 U, 再产生 Y, exp(λ)

exponential-uniform transformation.

$X \sim unif(0,1)$ $Y = -\log X \Rightarrow X = e^{-y}$ $\frac{dx}{dy} = -e^{-y}$.

$\therefore f_Y(y) = -e^{-y}$ $\sim \exp(1)$

example: if $U_j \sim unif(0,1)$ P248.

- $Y_j = -\lambda\log(U_j) \Rightarrow e^{-\frac{y_j}{\lambda}} = u_j \Rightarrow \frac{du_j}{dy_j} = -\frac{1}{\lambda}e^{-y_j/\lambda} \sim \exp(\lambda)$.
- $Y = -2(\log U_1 + \log U_2 + \dots + \log U_n)$

  $= \underbrace{-2\log U_1}_{\sim \exp(2)} + (-2)\log U_2 + \dots (-2)\log U_j$

  ∴ n 个 exp(2) 相加 $\sim Gamma(n, 2) \Rightarrow \chi^2(2n)$
- $Y = -\beta\sum_{j=1}^{a}\log(U_j) \sim Gamma(a, \beta)$ ← generalization.
- $Y = \frac{-\sum_{j=1}^{a}\log(U_j)}{-\sum_{j=1}^{a+b}\log(U_j)}$ → exp(1) $\frac{Gamma(a,1)}{Gamma(a,1)+Gamma(b,1)} = beta(a,b)$

  $X_1 \sim Gamma(a,1)$, $X_2 \sim Gamma(b,1)$ $\therefore f(x_1, x_2) = f(x_1)f(x_2)$

  $\therefore Y = \frac{X_1}{X_1+X_2}$ $Z = X_1$ ⇒ 得到 Y 的 dist Beta(a,b)

② acceptance/rejection method. (setup)

$\boxed{P(Y \le y) = P(V \le y \mid U < \frac{1}{c}f_Y(V))}$ ⇒ we see connections between $Y \le y$ & V.U

$$= \frac{P(V \le y, U < \frac{1}{c}f_Y(V))}{P(U < \frac{1}{c}f_Y(V))}$$

→ U.V indep Unif(0,1)

∴ we can use Unif U.V to generate Y.

$$= \frac{\int_0^y \int_0^{\frac{1}{c}f_Y(v)} du\,dv}{\int_0^1\int_0^{\frac{1}{c}f_Y(v)} du\,dv}$$

这里注意, 因为 U and V 都是 R.V., 所以要对两个 R.V 都积分.

$$= \frac{\int_0^y u\Big|_0^{\frac{1}{c}f_Y(v)} dv}{\int_0^{\frac{1}{c}f_Y(v)} du}$$

分母有: $\int_0^1\int_0^{\frac{1}{c}f_Y(v)} f_{U,V}(u,v)\,du\,dv$ 分子有: 已正确.

$\int_0^1\int_0^{\frac{1}{c}f_Y(v)} 1\,du\,dv$

$\int_0^1 \frac{1}{c}f_Y(v)\,dv$

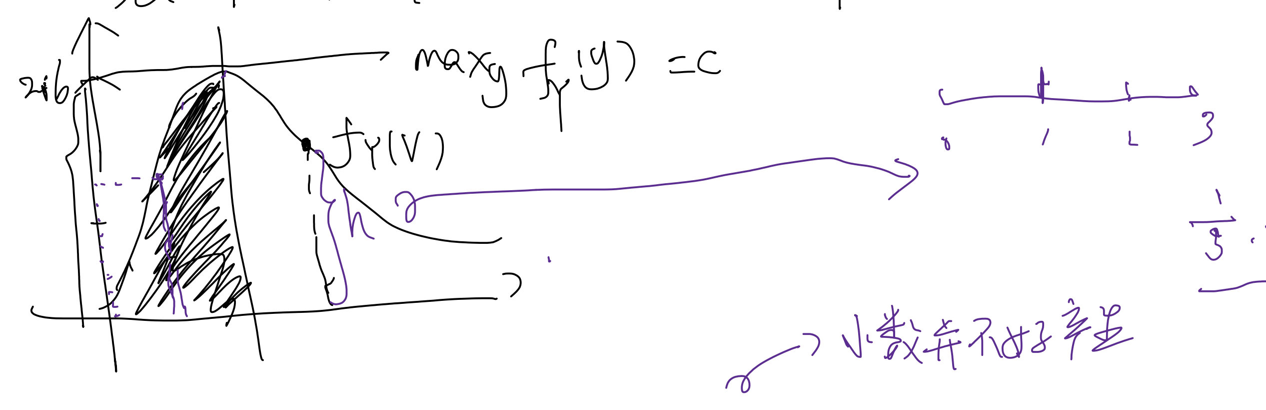
$= \frac{1}{c}$

$$= \frac{\int_0^y \frac{1}{c}f_Y(v)\,dv}{\frac{1}{c}f_Y(v)}$$

(a) generate $U, V \overset{iid}{\sim} Unif(0,1)$

(b) if $U < \frac{1}{c}f_Y(V)$ $c \ge \max_y f_Y(y)$

set $Y = V$. o/w, return to step (a)



to generate R.V. $Y \sim Beta(2.7, 6.3)$ → 小数并不好产生

(a) generate $U \sim Unif(0,1)$, $V \sim f_V$ Beta(2, 6)

(b) if $U < \frac{1}{M}f_Y(V)/f_V(V)$, set $Y = V$. o/w, return to step (a)

↳ the more Y and V "look like"

the more $U < \frac{1}{M}f_Y(V)/f_V(V)$, the more likely to accept.

$f(Y \le y) = f(V \le y \mid U < \frac{1}{M}f_Y(V)/f_V(V))$ 只有满足这个条件的才接受, 且这样接受的点则是 Y 的 PDF.

$$= \frac{f(V \le y, U \le \frac{1}{M}f_Y(V)/f_V(V))}{f(U < \frac{1}{M}f_Y(V)/f_V(V))}$$

→ 注意这里有两个 R.V.

$$= \frac{\int_{-\infty}^{y}\int_0^{\frac{1}{M}f_Y(v)/f_V(v)} 1\, f_V(v)\,du\,dv}{\int_{-\infty}^{+\infty}\int_0^{\frac{1}{M}f_Y(v)/f_V(v)} 1\, f_V(v)\,du\,dv}$$

∴ 要对所有的 R.V. 积分.

$= \int_{-\infty}^{y} f_Y(v)\,dv$ P253 SI.

$M = \sup_y f_Y(y)/f_V(y) < \infty$ ☆

③

$X = Unif(0,1) \to -\lambda\log(X) \sim Gamma)$

$X = Beta(a, 1)$ $-\log(X) \sim \exp(\frac{1}{a})$